

Non-Gaussianity Generated by the Entropic Mechanism in Bouncing Cosmologies Made Simple

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Non-gaussianity in the microwave background radiation is bound to play a key role in giving us clues about the physics of the very early universe. However, the associated calculations, at second and even third order in perturbation theory, tend to be complicated to the point of obscuring simple underlying physical processes. In this note, we present a simple analytic procedure for approximating the non-linearity parameters f_{NL} and g_{NL} for cyclic models in which the cosmological perturbations are generated via the entropic mechanism. Our approach is quick, physically transparent and agrees well with the results of numerical calculations.

Observations of the cosmic microwave background radiation are quickly becoming detailed enough that within the next few years we can hope to obtain highly informative limits on the bispectrum (and perhaps even the trispectrum) of primordial curvature perturbations [1]. In this respect the detection/non-detection of non-gaussianity will provide a powerful tool in discriminating between various theoretical models for the early universe. In simple inflationary models, the inflaton field is an almost free field, and correspondingly the curvature perturbations that these models generate are governed by very nearly gaussian statistics [2]. More complicated inflationary models, such as multi-field models, can produce pretty much any value for the so-called “local” non-linearity parameters f_{NL} and g_{NL} (corresponding to “squeezed” configurations in momentum space), which makes it difficult to predict a natural range [3]. So-called DBI models in which the inflaton possesses a non-canonical kinetic term lead to more distinct non-gaussian signals, involving quite different (“equilateral”) momentum configurations in their correlation functions [4], and multi-field DBI models can even produce significant contributions of both local and equilateral type simultaneously [5]. In ekpyrotic [6] and cyclic models [7], the cosmological perturbations are generated during a slowly contracting ekpyrotic phase with ultra-stiff equation of state $w_{ek} \gg 1$ (see [8] for a review). Such a phase can be modelled via scalar fields with steep negative potentials. The steepness of the potentials implies that these scalars are necessarily self-interacting, and this leads to natural values of the local non-linearity parameters that are in a range that will be

accessible to near-future observations [9–12]. Of course, this is why it is important to understand the physics that is responsible for the non-gaussian signals well, so that, in case of a detection, the consequences can be best appreciated.

Here, we will focus exclusively on cyclic models in which the cosmological perturbations are generated by the entropic mechanism, as this is currently the best understood mechanism for producing a nearly scale-invariant spectrum of curvature perturbations during a contracting phase [13]. Recently, both the associated bispectrum and trispectrum have been calculated numerically [11, 12]. These calculations are rather involved, and do not provide many clues about the final outcome. Hence it is desirable to develop analytic methods, even though they might only be approximate, to understand the physics of these calculations more thoroughly. Some headway in this direction was made by the authors in a recent publication dealing with the bispectrum calculation [14]. In this note we present a new and much simplified approach that can in fact be applied to the calculation of the trispectrum as easily as to that of the bispectrum.

The models under consideration can be described by gravity minimally coupled to two scalar fields with potentials. Nearly scale-invariant entropy perturbations are generated first, during a slowly contracting ekpyrotic phase that at the same time resolves the cosmological flatness puzzle. Subsequently, in the approach to the big crunch, the ekpyrotic potential becomes unimportant, and the universe enters a phase dominated by the kinetic energy of the scalar fields. During this phase, the entropy perturbations are converted into adiabatic curvature perturbations with the same spectrum, and these curvature perturbations form the seeds of the large-scale structure during the subsequent expanding phase. We will discuss the kinetic conversion phase in more detail below.

We adopt the following parametrization of the potential during the ekpyrotic phase:

$$V_{ek} = -V_0 e^{\sqrt{2\epsilon}\sigma} \left[1 + \epsilon s^2 + \frac{\kappa_3}{3!} \epsilon^{3/2} s^3 + \frac{\kappa_4}{4!} \epsilon^2 s^4 + \dots \right], \quad (1)$$

where we expect $\kappa_3, \kappa_4 \sim \mathcal{O}(1)$ and where $\epsilon \sim \mathcal{O}(10^2)$ is related to the ekpyrotic equation of state w_{ek} via $\epsilon = 3(1 + w_{ek})/2$. We use σ to denote the adiabatic direction, *i.e.* the direction tangent to the scalar field space trajectory, and s to denote the “entropy” direction, *i.e.* the direction perpendicular to the background trajectory (note that the fields σ and s are thus defined such that the coordinate system they imply moves along with the background trajectory [15]); see Fig. 1. The ekpyrotic potential is tachyonic in the entropy direction, and this instability causes the entropy perturbations to grow [13]. Moreover, this instability has the consequence that the global structure becomes a “phoenix” universe, in which the universe loses most of space to black holes

at the end of each cycle, while the regions that survive the big bang are aided by the dark energy to grow into vast new habitable regions – this was discussed in some detail in [16].

During the ekpyrotic phase, it is straightforward to solve for the entropy perturbation, with the result that [12]

$$\delta s = \delta s_L + s_2 \delta s_L^2 + s_3 \delta s_L^3, \quad (2)$$

with the linear, gaussian part δs_L being inversely proportional to time t (defined below in Eq. (9)); the coefficients s_2 and s_3 are given in terms of the parameters of the potential by

$$s_2 = \frac{\kappa_3 \sqrt{\epsilon}}{8}, \quad (3)$$

$$s_3 = \left(\frac{\kappa_4}{60} + \frac{\kappa_3^2}{80} - \frac{2}{5} \right) \epsilon. \quad (4)$$

The local non-linearity parameters f_{NL} and g_{NL} can be defined via an expansion of the curvature perturbation ζ in terms of its linear, gaussian part ζ_L ,

$$\zeta = \zeta_L + \frac{3}{5} f_{NL} \zeta_L^2 + \frac{9}{25} g_{NL} \zeta_L^3. \quad (5)$$

Then, it was found numerically that, for conversions lasting on the order of one e-fold of contraction of the scale factor, the non-linearity parameters can be well fitted by the simple formulae [11, 12]

$$f_{NL} \approx 12s_2 + 5 = \frac{3}{2} \kappa_3 \sqrt{\epsilon} + 5 \quad (6)$$

$$g_{NL} \approx 100s_3 = 100 \left(\frac{\kappa_4}{60} + \frac{\kappa_3^2}{80} - \frac{2}{5} \right) \epsilon. \quad (7)$$

The simplicity of the end result (given the complications of the third order perturbation equations involved) suggests that there ought to be a more straightforward way to obtain it. In fact, the physics of the kinetic phase (which follows the ekpyrotic phase, and during which the conversion takes place) is really quite simple, and moreover, except for the fact that its initial conditions involve the entropy perturbation δs , the kinetic phase has no memory of the details of the ekpyrotic phase. In particular, only the total δs in (2) matters, and the way we choose to decompose it into linear, second- and third-order parts is irrelevant at this point. This realization is the first ingredient of our calculation.

The second is a compact and very useful expression for the evolution of the curvature perturbation ζ on large scales and in comoving gauge [10, 17]:

$$\dot{\zeta} = \frac{2\bar{H}\delta V}{\dot{\sigma}^2 - 2\delta V}, \quad (8)$$

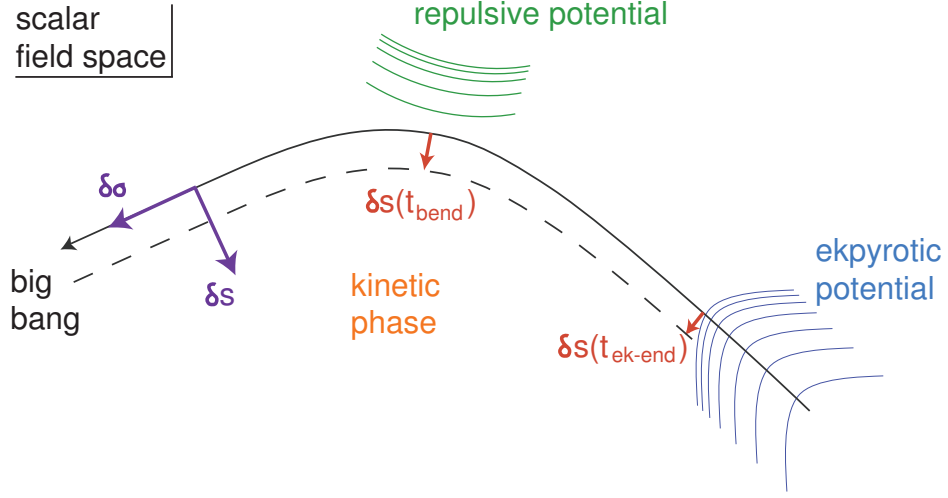


FIG. 1: After the ekpyrotic phase, the trajectory in scalar field space enters the kinetic phase and bends - this bending is described by the existence of an effective repulsive potential (the potentials are indicated by their contour lines). A trajectory adjacent to the background evolution can be characterized by the entropy perturbation $\delta s(t_{\text{ek-end}})$ at the end of the ekpyrotic phase, leading to a corresponding off-set $\delta s(t_{\text{bend}})$, or equivalently $\delta V(t_{\text{bend}})$, at the time of bending.

where a dot denotes a derivative w.r.t. time t , $H \equiv \dot{a}/a$ is the Hubble parameter and a the scale factor, $\delta V \equiv V(t, x^i) - \bar{V}(t)$ and a bar denotes a background quantity. This equation is exact in the limit where spatial gradients can be neglected, and can thus be expanded up to the desired order in perturbation theory if required. First, let us present its derivation [17]: considering only very large scales, we can write the metric as

$$ds^2 = -dt^2 + a^2(t)e^{2\zeta(t, x^i)}dx^i dx_i, \quad (9)$$

where all the inhomogeneities are in ζ . This defines ζ to all orders in the long-wavelength limit. Then the equation of continuity reads

$$\dot{\rho} + 3(H + \dot{\zeta})(\rho + P) = 0, \quad (10)$$

where ρ is the (scalar) matter energy density, and P its pressure. But we're interested in the curvature perturbation on surfaces of uniform energy density, so $\rho = \bar{\rho}$ (and hence also $H = \bar{H}$). And since $\bar{\rho}$ satisfies $\dot{\bar{\rho}} + 3\bar{H}(\bar{\rho} + \bar{P}) = 0$, we immediately obtain

$$\dot{\zeta} = -\bar{H} \frac{\delta P}{\bar{\rho} + \bar{P} + \delta P}. \quad (11)$$

Now, since we choose to consider hypersurfaces on which $\delta\rho = 0$, we obtain the relations $\delta(\dot{\sigma}^2) = -2\delta V$ and thus $\delta P = -2\delta V$. Plugging these relations into (11) then yields our desired result, Eq. (8).

Incidentally, by expanding Eq. (8), it is also possible to show that it is equivalent to the third-order equation derived in [12] using the covariant formalism. The crucial thing is to keep in mind the definitions of the adiabatic perturbation $\delta\sigma$ and the entropic one δs at higher orders, provided in [15] and [12]. In particular, we have (from hereon we drop the bar on background quantities - this should not lead to any confusion)

$$\begin{aligned}\delta V = & V_{,s}\delta s + V_{,\sigma}\delta\sigma \\ & + V_{,s}[\delta s^{(2)} + \frac{1}{\dot{\sigma}}\delta\sigma\dot{\delta s} + \frac{\dot{\theta}}{2\dot{\sigma}}(\delta\sigma)^2] + V_{,\sigma}[\delta\sigma^{(2)} - \frac{1}{2\dot{\sigma}}\delta s\dot{\delta s}] \\ & + \frac{1}{2}V_{,ss}(\delta s)^2 + V_{,s\sigma}\delta s\delta\sigma + \frac{1}{2}V_{,\sigma\sigma}(\delta\sigma)^2 \\ & + \dots\end{aligned}\tag{12}$$

Up to third order and in comoving gauge ($\delta\sigma = \delta\sigma^{(2)} = \delta\sigma^{(3)} = 0$) we then get

$$\begin{aligned}\delta V = & V_{,s}\delta s \\ & + V_{,s}\delta s^{(2)} + V_{,\sigma}(-\frac{1}{2\dot{\sigma}}\delta s\dot{\delta s}) + \frac{1}{2}V_{,ss}(\delta s)^2 \\ & + V_{,s}\delta s^{(3)} - V_{,\sigma}[\frac{1}{2\dot{\sigma}}(\delta s\delta s^{(2)})] + \frac{\dot{\theta}}{6\dot{\sigma}^2}(\delta s)^2\dot{\delta s} + V_{,ss}\delta s\delta s^{(2)} - \frac{V_{,s\sigma}}{2\dot{\sigma}}(\delta s)^2\dot{\delta s} + \frac{1}{6}V_{,sss}(\delta s)^3,\end{aligned}\tag{13}$$

and by expanding (8) we obtain

$$\begin{aligned}\dot{\zeta} = & \frac{2H}{\dot{\sigma}^2}V_{,s}\delta s \\ & + \frac{2H}{\dot{\sigma}^2}[V_{,s}\delta s^{(2)} - \frac{1}{2\dot{\sigma}}V_{,\sigma}\delta s\dot{\delta s} + \frac{1}{2}V_{,ss}(\delta s)^2] + \frac{4H}{\dot{\sigma}^4}V_{,s}^2(\delta s)^2 \\ & + \frac{2H}{\dot{\sigma}^2}[V_{,s}\delta s^{(3)} - \frac{1}{2\dot{\sigma}}V_{,\sigma}(\delta s\delta s^{(2)}) - \frac{\dot{\theta}}{6\dot{\sigma}^2}V_{,\sigma}(\delta s)^2\dot{\delta s} + V_{,ss}\delta s\delta s^{(2)} - \frac{V_{,s\sigma}}{2\dot{\sigma}}(\delta s)^2\dot{\delta s} + \frac{1}{6}V_{,sss}(\delta s)^3] \\ & + \frac{8H}{\dot{\sigma}^4}[V_{,s}^2\delta s\delta s^{(2)} - \frac{1}{2\dot{\sigma}}V_{,s}V_{,\sigma}(\delta s)^2\dot{\delta s} + \frac{1}{2}V_{,s}V_{,ss}(\delta s)^3] + \frac{8H}{\dot{\sigma}^6}V_{,s}^3(\delta s)^3,\end{aligned}\tag{14}$$

which agrees precisely with the equation derived in [12]. Having shown this equivalence, we will now stick with the simple and compact form (8).

We are assuming that, during the kinetic phase, the trajectory in scalar field space contains a bend, and this bend is what causes the entropy perturbations to source the curvature perturbations. In cyclic models embedded into heterotic M-theory, such a bend occurs naturally because the scalar field space contains a boundary which effectively acts as a repulsive potential (we refer the reader to Ref. [18] for details). However, we are simply citing this example as a concrete realization.

Our calculation applies to all cases where there is a bend in the trajectory, although for extreme (and unnatural) cases where the bending angle is close to 0° or 180° some of our approximations below might break down. We are assuming that this bend can be described as being caused by a monotonic repulsive potential, as depicted in Fig. 1.

Now, the third and last ingredient of our calculation is the simple relationship between δV and δs during the conversion process. During the ekpyrotic phase, the curvature perturbation picks up a blue spectrum [19] and is hence completely negligible on large scales. To be precise, since $\delta V \neq 0$ during ekpyrosis, there is already some conversion of entropy into curvature perturbations occurring at this stage. However, this contribution is entirely negligible compared to the subsequent conversion (see [12, 14] and note that since $V_{,s} = 0$ during ekpyrosis, δV starts out at subleading order), and hence we can take $\zeta(t_{ek-end}) \approx 0$ where t_{ek-end} denotes the time at the end of the ekpyrotic phase, or equivalently, at the start of the kinetic phase. Moreover, as we will see below, at the end of the conversion process ζ is still significantly smaller than δs , and hence, during the conversion process, we can take the potential to depend only on δs . And since the repulsive potential is monotonic, and we are interested in small departures $\delta s \ll 1$ from the background trajectory, it is intuitively clear that δV is directly proportional to δs during the bending. A numerical calculation readily confirms this simple relationship.

During the conversion, the effect of the repulsive potential is to cause the entropy perturbation to behave approximately sinusoidally, independently of the precise functional form of the potential (this was shown analytically in [14]):

$$\delta s \approx \cos[\omega(t - t_c)]\delta s(t_c), \quad (15)$$

where t_c denotes the time at which the conversion starts. Moreover, the precise value of $\delta s(t_c)$ is unimportant for the present calculation. The frequency ω is of $\mathcal{O}(1/\Delta t)$, where Δt is the duration of the conversion; the more careful analysis presented in [14] leads to $\omega \approx 2.5/\Delta t$. Another useful quantity is the rate of change of the angle of the trajectory in scalar field space [20]

$$\dot{\theta} \equiv -\frac{V_{,s}}{\dot{\sigma}} \approx \frac{1}{\Delta t}. \quad (16)$$

Also, the scale factor and the scalar field velocity along the background trajectory are rather unaffected by the presence of the repulsive potential, so that they simply assume the values they would in the absence of any potential

$$H = \frac{1}{3t}, \quad \dot{\sigma} = \frac{-\sqrt{2}}{\sqrt{3t}}. \quad (17)$$

As we will confirm below, during the conversion process $\delta V \ll \dot{\sigma}^2$, so that Eq. (8) simplifies further to

$$\dot{\zeta} \approx \frac{2\bar{H}}{\dot{\sigma}^2} \delta V. \quad (18)$$

Then, at linear order, we immediately obtain

$$\zeta_L = \int_{bend} -\frac{2H}{\dot{\sigma}} \dot{\theta} \delta s_L \quad (19)$$

$$\approx \sqrt{\frac{2}{3}} \frac{\dot{\theta}}{\omega} \sin(\omega \Delta t) \delta s(t_c) \quad (20)$$

$$\approx \frac{1}{5} \delta s(t_c). \quad (21)$$

But, as argued above, δs as a whole must behave approximately in this way during the conversion phase, and subsequently analogous relationships hold at higher orders too:

$$\zeta^{(2)} \approx \frac{1}{5} s_2 \delta s_L^2 \quad \zeta^{(3)} \approx \frac{1}{5} s_3 \delta s_L^3. \quad (22)$$

These expressions immediately allow us to calculate the non-linearity parameters

$$f_{NL} \equiv \frac{5}{3} \frac{\zeta^{(2)}}{\zeta_L^2} \approx \frac{5}{3} s_2 \approx 8 s_2 \quad (23)$$

$$g_{NL} \equiv \frac{25}{9} \frac{\zeta^{(3)}}{\zeta_L^3} \approx \frac{25}{9} s_3 \approx 70 s_3. \quad (24)$$

Thus, without much work at all, and to better accuracy than a factor of 2, we recover the numerically (and laboriously) obtained fitting formulae in Eqs. (6)-(7) above.

Before discussing this result, let us briefly pause to verify the approximation made in obtaining Eq. (18): during the kinetic phase, we can rewrite (8) as

$$\dot{\zeta} = \frac{t \delta V}{1 - 3t^2 \delta V}. \quad (25)$$

The approximation made above consists in writing $\dot{\zeta} \approx t \delta V$ and this leads to $\zeta \approx \frac{1}{2} t_{bend}^2 \delta V(t_{bend})$. But we know that by the end of the conversion process $\zeta \approx \frac{1}{5} \delta s$ and hence we find that

$$3t_{bend}^2 \delta V \approx \delta s \ll 1, \quad (26)$$

which shows that the approximation is self-consistent and confirms the validity of (18).

So what does our result tell us? The main point is that due to the simplicity of the kinetic phase, the non-linearity that was present in the entropy perturbation gets transferred straightforwardly to the non-linearity in the curvature perturbation. Our calculation therefore explains why there

are no significant additional constant terms in (6) or constants and κ_3 -dependent terms in (7); *a priori*, there was no reason for such terms to be absent.

Moreover, the overall magnitude of f_{NL} and g_{NL} is set solely by the efficiency of the conversion process, as expressed by the relationship between δs_L and ζ_L in Eq. (21). The fact that no additional parameter enters into Eq. (21) has important consequences for observations, as it determines the scaling of the non-linearity parameters with the equation of state parameter ϵ , as expressed in Eqs. (6) and (7). A natural value for ϵ would be about 50, so that we can expect f_{NL} to be of order a few tens, with the sign typically determined by the sign of κ_3 , and g_{NL} to be of order a few thousand and typically negative in sign. These values represent the natural values predicted by models making use of the entropic mechanism. They comfortably fit current observational bounds [21, 22] while being detectable by near-future observations. It is useful to contrast these values with those predicted by “new ekpyrotic” models [23, 24] where the entropy perturbations are converted into curvature perturbations directly during the ekpyrotic phase. For this variant conversion process, the dependence on the equation of state ϵ is more pronounced, with $f_{NL} \propto \epsilon$ and $g_{NL} \propto \epsilon^2$ [9, 10, 12]. In addition, f_{NL} is predicted to take a negative value and g_{NL} a positive one; the magnitude and sign do not fit well with current observations.

Finally, we note that, in spirit, our approach is somewhat reminiscent of the δN formalism [25, 26]. However, a full δN calculation spanning both the generation and conversion of the cosmological perturbations is made difficult here because of the transition between the ekpyrotic and kinetic phases; note, in particular, that in going from the ekpyrotic to the kinetic phase, the equation of state drops drastically from $w_{ek} \gg 1$ to $w_{kin} \approx 1$ (by contrast, the δN formalism is well adapted to new ekpyrotic models where the generation and conversion both take place during the ekpyrotic phase [9]). In fact, it is precisely the disconnectedness between the two phases that allows our method to work so well, and we would expect it to be applicable more generally to cases where the physical properties of the phases of generation and conversion differ substantially. The separation between the two phases allows for a two-stage approach in which we first solve for the entropy perturbation during the ekpyrotic phase, and then use this as input for calculating ζ or, equivalently δN , by perturbing around the background trajectory during the kinetic phase. The simple, yet non-perturbative, Eq. (8) then reveals its full effectiveness by yielding the result in just a few lines of derivation. In this way, we have found a quick and rather accurate way of understanding non-gaussianity in two-field cyclic models of the universe.

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